On Self-Tiling Tile Sets

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The first telephone was invented in 1853. However, the device never really caught on until the invention of the *second* telephone some time later.

—Spike Milligan [author's paraphrase]

An animal familiar to polyform specialists is the *rep-tile*, or self-replicating tile, first introduced by Solomon Golomb [**1**]. It is a planar shape that can be tiled with smaller replicas of itself. Hitherto, rep-tiles have usually been regarded as singular creatures exhibiting few links with the rest of the animal kingdom. Here I suggest that with only a small change in perspective they can be identified as close cousins of a previously unexplored object that I call a *self-tiling tile set*.

By a self-tiling tile set of order n, I mean a set of n distinct (non-similar) planar shapes, each of which can be tiled with smaller replicas of the complete set of n shapes. We require that the scaling factor be the same for each piece. FIGURE 1 shows an example for $n = 4$ formed of hexominoes. Note that some pieces are flipped.

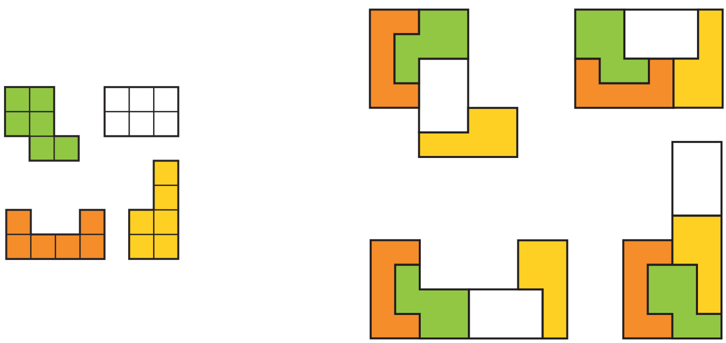

Figure 1 A self-tiling tile set of order 4 using hexominoes

Here the four hexomino-shaped areas at right are each tiled by the four similar hexominoes at left. Or, to look at the same property through the other end of the telescope, the four shapes shown, small or large, can each be dissected into four distinct pieces that are smaller duplicates of the same four figures. Clearly, the twin actions of forming still larger and larger copies (known as *inflation*), or still smaller and smaller dissections (*deflation*), can be repeated indefinitely. Here we consider sets using flat pieces only, but the extension to 3-D pieces is obvious.

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Note that, since the n members of any such set are each tiled by the same n pieces, their areas are necessarily equal. Also, since the area of any compound copy formed by combining the n pieces is n times that of its smaller version, the increase in *scale* is $\sqrt{n}$. For example, in FIGURE 1 the area increase is 4 times, indicating a scale factor of $\sqrt{4} = 2$, meaning that larger pieces are twice the size of smaller. Hence, if a set is to use n *polyforms*, which have integral sizes, then the compound shapes produced by combining n of them will also have integral sizes, showing that $\sqrt{n}$ must be an integer. A self-tiling tile set of polyforms using a non-square number of pieces is thus impossible, so that beyond $n = 4$ the next possibility becomes $n = 9$.

How do we go about finding such a set? Although straightforward in principle, a search by computer turns out to be quite demanding. In examining the hexominoes, for example, of which there are 35 distinct shapes, I began with a program that identifies every possible tiling of a *bighex* (a doubled-in-size-hexomino-shaped area) by four distinct hexominoes. The latter are each represented by a distinct integer, a *hexnum*, so that the program's output is in the form of a list of *quads*, each consisting of 4 hexnums representing a distinct tiling of the bighex. Repeating this process for every bighex in turn, I ended up with 35 lists of quads. Many of these quads occur in more than one of the 35 sets, since the same 4 pieces often tile different bighexes. Taking each quad in turn, a second program then constructed a list of those bighexes it tiles (the latter again represented by their corresponding hexnums), followed by a search for any quad, all 4 of whose members appear among its associated list of bighexnums. Only one such quad was found; it is the one represented by FIGURE 1. Alas, sets for which $n > 4$, involving still larger bighexes, were beyond the scope of the program and thus remain unexamined. My grateful thanks are due to polyforms expert Pat Hamlyn, who was kind enough to supply me with the software used in the researches here described.

Stimulated by this single find, I turned next to the heptominoes, of which there are 108. Patiently entering data to the program so as to define the shapes to be tiled was a lengthy task, requiring some two days' determined effort. It came as quite a blow when the program discovered no solutions whatever. Worse yet, subsequent experiments yielded no further solutions for any polyominoes smaller than octominoes. There are 369 octominoes, a forbidding total in view of the work their examination entails. When at length I did launch into an exploration of the octominoes, I found myself surveying a staggeringly different world.

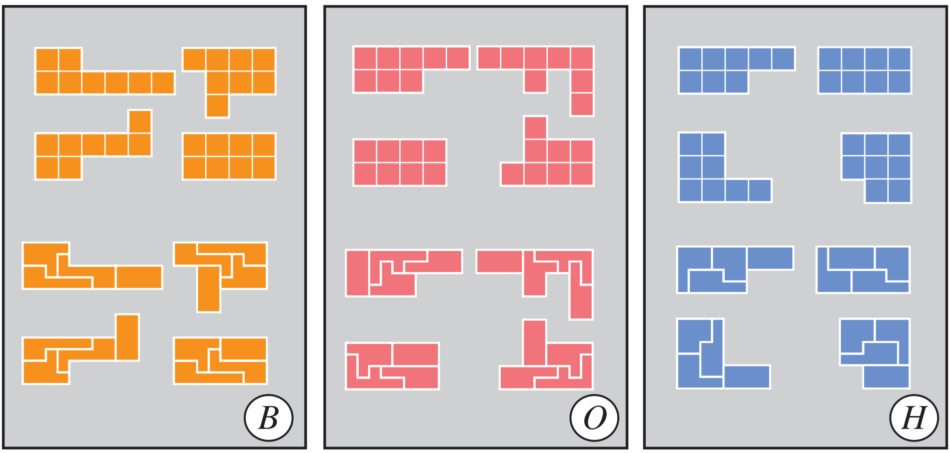


Figure 2 Three of the seven self-tiling tile sets using 4 octominoes

The octomino network for $n = 4$

To begin with, there exist as many as *seven* distinct self-tiling tile sets using four octominoes. Three of them are shown in FIGURE 2, labeled B, O, and H; the remaining four can be found in FIGURE 8. These were pleasing finds, but on running an eye over the computer output data, I soon noticed signs of a phenomenon already foreseen as a possibility from the start, namely, the existence of loops, which is to say, of closed chains of sets, each of which tiles its successor. After making a few changes to the program, I settled down to study the loops in detail. FIGURE 3, for example, records one of the length-2 loops found, which is to say, a pair of *co-* or *mutually*-tiling sets. The two sets of four pieces shown are distinct, although one octomino, the rectangle, is common to both. The fact that the same piece occurs in both sets is perhaps a pity, but acceptable so long as our requirement is merely that *sets* be distinct.

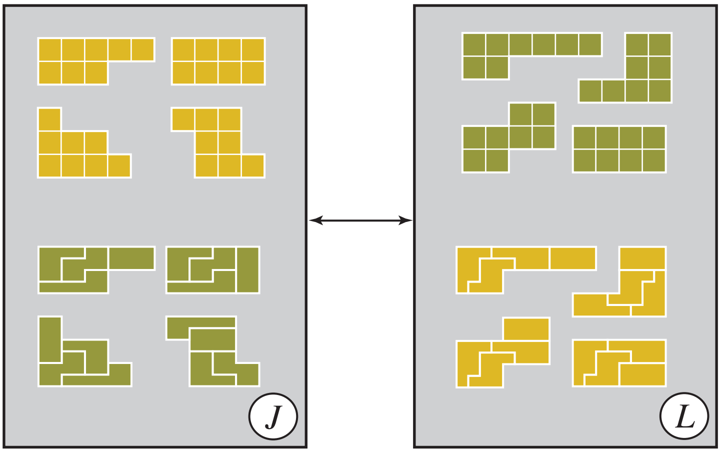
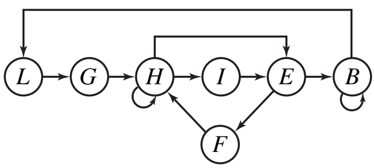


Figure 3 A pair of mutually tiling sets or a closed loop of length 2

Curiously, the same rectangular octomino appears in every one of the sets met with below. In FIGURE 3, the four green octominoes tile (a larger replica of) the yellow, that in turn tile (a larger replica of) the green. From here on it will be convenient to take the phrase "a larger replica of" as understood, and to speak simply of one set tiling another. Likewise, to save space, tiled shapes in the sets pictured are also shown non-enlarged. In the figures to follow, an arrow pointing from set S_1 to set S_2 indicates that S_1 tiles S_2, a relation we can represent symbolically by a directed graph thus:

The need for some such notation made itself felt, as the loops emerging from the data became not merely longer, but even intertwined with each other in complicated ways. For example, FIGURES 4 and 5 show two of the loops found, one of length 3 and one of length 6, respectively. But sets E and H appear in both loops, while sets B and H are among the self-tilers in FIGURE 2, a tangle that is nevertheless neatly captured by the following graph:



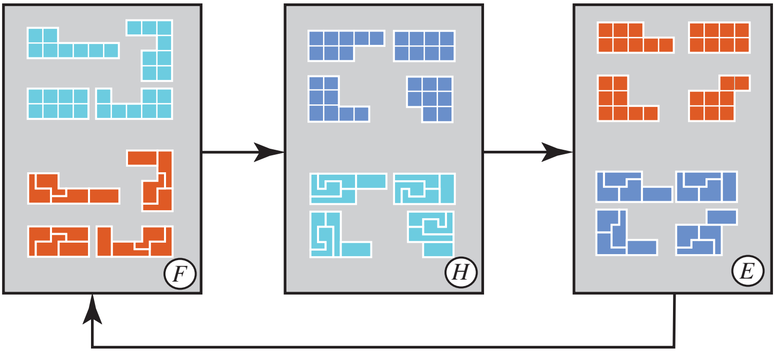


Figure 4 A loop of length 3

Armed with this simple device, initial attempts to map out the complete octomino network soon began to reveal an extraordinary complexity. It was a complexity that quite overwhelmed my elementary methods. Thus far, loops had been traced by hand. Before long it became clear that such an approach could never identify them all. FIGURE 7, for example, itself a veritable rat's nest of interconnections, records a mere fragment of the whole. It is a graph of those loops associated with a restricted subset of the octominoes, namely those small enough to fit within a 12×6 rectangle. Even so, FIGURE 7 is of illustrative use. Observe that its lettered nodes correspond to the lettered sets of same colour appearing in other figures. Inspection will reveal 9 cases of lines with double-headed arrows, indicating 9 loops of length 2. Interested readers may enjoy verifying its structure through tracing the remaining loops, several more of which can be found for all lengths up to a maximum of 8. To this end, for completeness, the six sets A, C, D, K, M, and N, which do not appear in other figures, are included in FIGURE 6. The task of discovering how the pieces of one set fit together so as to tile those of another must then be arrived at by trial and error. Note that while set

Figure 5 A loop of length 6

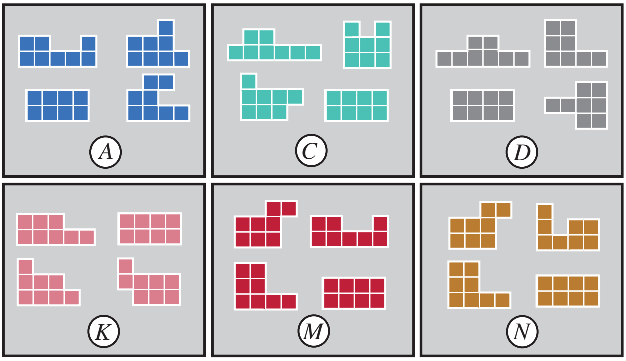


Figure 6 Six remaining sets that (together with those shown in the other figures) enable a complete verification of FIGURE 7

O in FIGURE 2 tiles sets A, B, D, F, and L, as well as tiling itself, it does not form part of any other loop, for which reason it is not included in FIGURE 7.

Meanwhile, the problem of unraveling the complete network had defeated my every effort. Small wonder then that I sought outside help. In the aftermath, warm thanks are due to my ex-colleague Henk Schotel, formerly a cognitive scientist at the Radboud University in The Netherlands, whose application of Frank Meyer's Java program [**2**] implementing Tarjan's well-known graph-searching algorithm [**3, 4**], provided a perfect tool for the job. Almost all the credit for securing a complete description of the network belongs to Henk Schotel, without whose generous help and expertise this account would be much the poorer.

Nevertheless, it is a matter of regret that the network brought to light by Schotel is simply too large, and its interconnections too dense to be captured in a legible

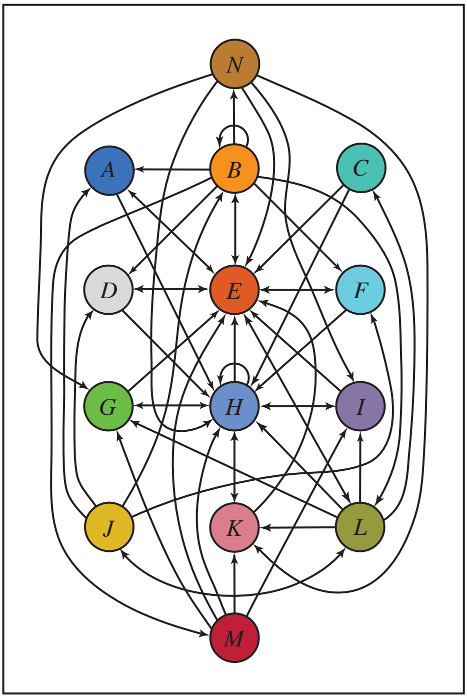


Figure 7 A small fragment of the complete octomino network

graph, containing as it does 62 nodes joined by 461 links. I have already mentioned that this includes seven self-tilers. Beyond these, however, there appears an astounding plethora of loops, ranging in length from 1 up to 14. TABLE 1 records their precise frequencies, several of them in the hundreds of thousands. The total number of loops found is thus not far short of one and a half million. Alas, even to provide the very minimum of information necessary to enable readers to reconstruct the network for themselves would again require overmuch space here. For example, among other data required, the adjacency matrix involved is of size 62 × 62. In consequence, full details together with explanatory notes have been made available on the author's website, http://LeeSallows.com.

TABLE 1: Loop lengths and their frequency of occurrence in the octomino network

Loop length	Number of loops
1	7
2	31
3	162
4	807
5	3,330
6	11,413
7	32,683
8	78,384
9	158,040
10	260,408
11	334,896
12	316,800
13	186,240
14	46,080
	1,429,281

By means of this, polyform enthusiasts curious to explore the octomino labyrinth for themselves can be sure of encountering some arresting structures. One such is what I call the 'sunburst', so-called because its corresponding digraph consists of a central node (set E in FIGURE 5) surrounded by 14 satellite nodes to which it is linked by 14 'rays' of double-headed arrows. Or in other words, set E is the common member of an astonishing 14 separate loops of length 2. Once again, limitations of space make it impractical to depict all 14 of these sets here. Set E is in fact by far the 'busiest' node in the network, occurring as it does in over 99.9% of all loops. Still more striking is the 'quintet', an amazing family of five sets pictured in FIGURE 8. Every n-tuple of sets forms a bidirectional loop of length n. That is, every set is a self-tiler, every pair of sets is a loop

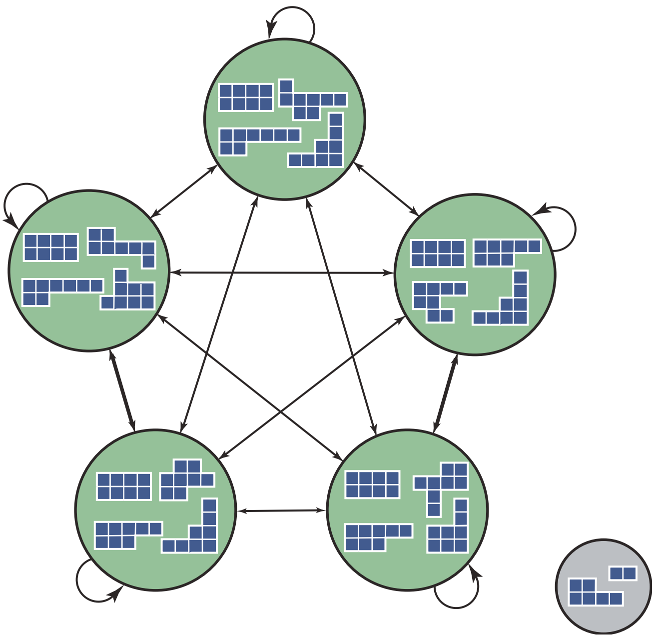


Figure 8 "The Quintet": Every set is a self-tiler; every pair of sets tile each other.

of length 2, every triad of sets is a loop of length 3, and so on. As a result, the 4 pieces in any one of the 5 sets will assemble to yield a twice-sized copy of any chosen piece. It is easy to explain how this adaptability comes about. Examination will confirm that every one of the 11 distinct octominoes appearing can be formed by distinct juxtapositions of just two component shapes: the hexomino and domino shown below right in FIGURE 8. Less obvious is that the three non-rectangle pieces in each set will themselves combine to produce an enlarged copy of this hexomino. But this means that the latter can then be juxtaposed with the remaining rectangle (= enlarged domino) so as to yield larger duplicates of any of the original 11 pieces. Incidentally, with just three exceptions, the rectangular octomino is found in all 62 sets of the complete octomino network.

A point worth noting here is that, although structures such as the sunburst and quintet are 'there' in the network, they still have to be sought for and identified amid the tightly interlacing reticulation. The process of finding them is thus not unlike that of the sculptor, whose task is to chip away stone obscuring the image hidden within the block. Recall too that in practice we are not looking at a real digraph such as FIGURE 7, but at computer output in the shape of lists of numbers corresponding to such a graph. There is thus plenty of scope for serious detective work in the process of tracking down patterns of interest concealed within the tracery. So much then for a brief account of the reflexive tiling properties of the octominoes for the case $n = 4$.

Beyond polyforms

Thus far we have made no more than an initial foray into an as yet largely unexplored field. Doubtless future explorers will extend these researches to still other and larger polyforms.

However, the pieces used in a self-tiling tile set do not have to be polyforms; they may be planar shapes of any kind, in which case n, the number of pieces in the set, need not be restricted to *square* numbers. But in that case, what of the *smallest* possible self-tiling tile set, which is of order 2, or one using just two pieces? The question thus prompted may be stated as follows:

Do there exist two non-similar planar figures, A and B, such that each can be dissected into two smaller figures, A' and B', where A' is similar to A and B' is similar to B?

It is a problem that occupied me for weeks before finally hitting on a whole class of solutions using two triangles. We shall look at them below. In the meantime, the

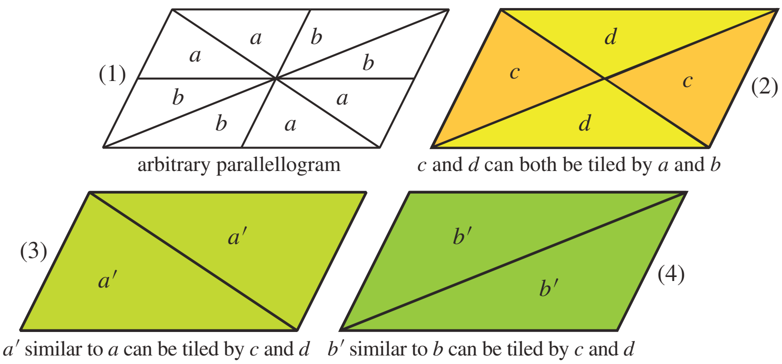


Figure 9 The parallelogram method for producing a pair of co-tiling sets composed of two triangles

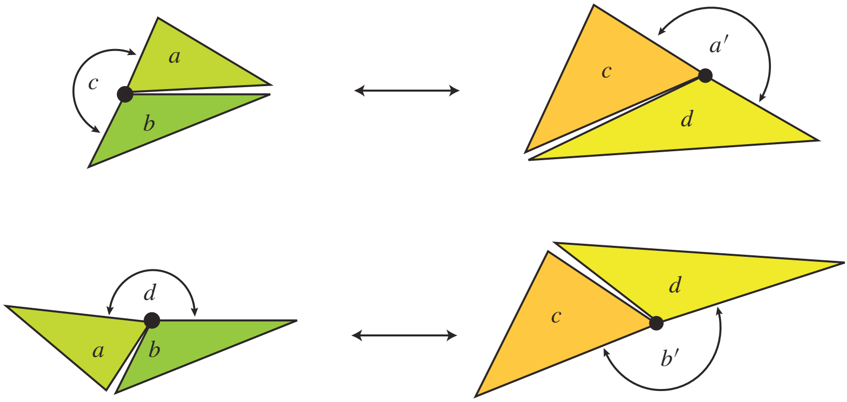


Figure 10 Above: triangles *a* and *b* tile *c*, while *c* and *d* tile *a*′, which is similar to *a*. Below: triangles *a* and *b* tile *d*, while *c* and *d* tile *b*′, which is similar to *b*.

finding is more informatively introduced by considering first a solution to a related problem, which is that of discovering a *pair* of mutually-tiling sets, each composed of two pieces.

Consider FIGURE 9(1), which depicts an arbitrary parallelogram divided into 8 triangles by the four bisectors shown. The triangles are of two distinct shapes, labeled a and b. FIGURE 9(2) identifies two distinctly shaped regions c and d, each of which corresponds to a distinct union of a and b. That is, both c and d can be tiled with the triangles a and b. In similar fashion, triangle a' in FIGURE 9(3) is a union of c and d, just as triangle b' in FIGURE 9(4) is a different union of c and d, so that each of the triangles a' and b' can be tiled by the triangles c and d. But a' is similar to a, and b' is similar to b, which completes a loop of length 2. In short, triangles a and b tile both c and d, which in turn tile (larger copies of) a and b. Note that since we may start with any parallelogram, the variety of triangle shapes in the sets resulting can be varied continuously. Less obvious is that the dissections of triangles a', b', c, and d that result are all *hinged* dissections, as illustrated in FIGURE 10. Here I have omitted relative angles and side lengths, which are easily derivable from the initial parallelogram chosen.

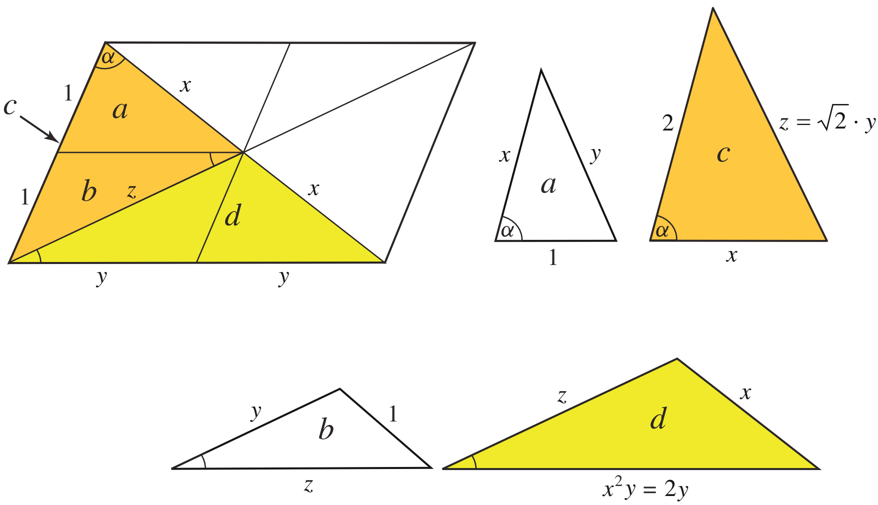


Figure 11 If triangles *a* and *b* are similar to *c* and *d*, then *x* must equal $\sqrt{2}$.

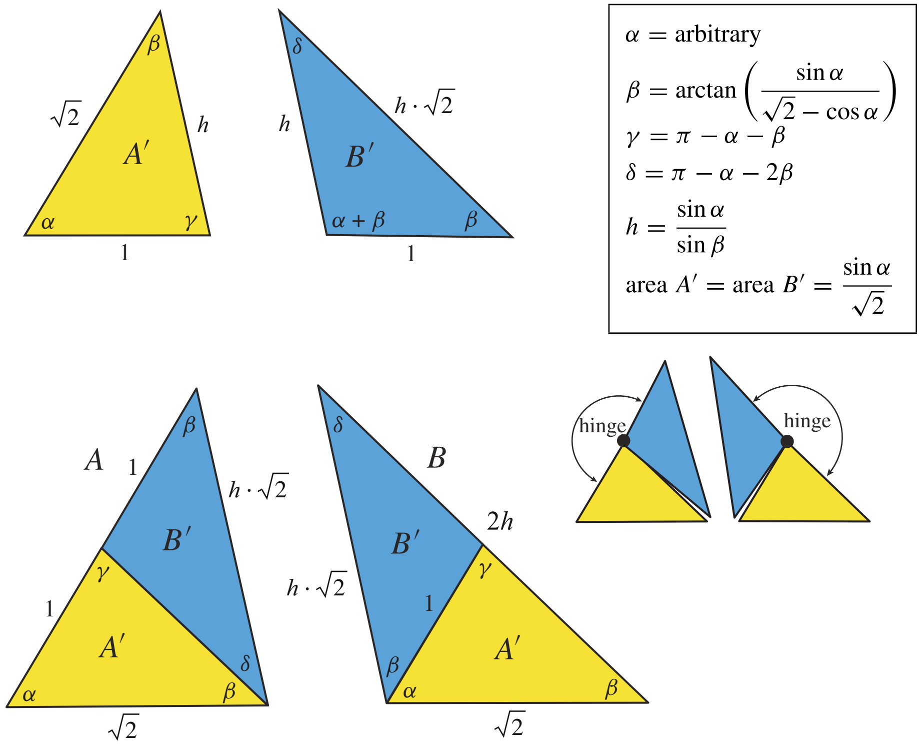


Figure 12 A class of self-tiling tile sets of order 2: A and B are similar to A' and B', respectively.

This completes a look at the "parallelogram method" for producing a pair of co-tilers comprising two triangles in each set. A couple of special cases to consider are when the parallelogram employed is either a rectangle or a square, with the result that triangles a and b become congruent. A more interesting case occurs when the parallelogram chosen is such as to make triangles c and d *similar* to a and b, respectively. The outcome is then that triangles a and b are able to tile larger versions of *themselves*, which is exactly the property at the focus of the question above.

A family of order-2 sets

Under what conditions will triangles a and b be similar to c and d, respectively? FIGURE 11 reproduces the parallelogram of FIGURE 9(1). Labels identify edge lengths in the various triangles. We can assume without losing generality that the short side of triangle a is of length 1. Suppose now a and c are similar; each side of a thus has its corresponding side in c. The angle α appears in both, and both share a side of length x. But triangle c is larger than a, so that the side of length x in one cannot correspond to the side of length x in the other. In light of these points, to the right, triangles a and c have been reproduced (rotated and reflected), but now oriented so that their common angle, α, appears below left in both, with corresponding sides now in corresponding positions. The difference in base lengths shows that the scale factor must be x, so that on comparing the left hand edges of a and c, we find $x^2 = 2$. Hence in triangle c, $z = x \cdot y = \sqrt{2} \cdot y$. Analogously, in FIGURE 11, triangles b and d are likewise oriented, their common angle and resulting edge lengths confirming that, like a and c, triangles b and d are similar exactly when $x = \sqrt{2}$.

$$\alpha = \arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \arccos\frac{1}{\sqrt{3}}$$
$$\beta = \arcsin\left(\frac{1}{\sqrt{3}}\right) = \arccos\frac{\sqrt{2}}{\sqrt{3}}$$

$$\alpha = \arccos\left(\frac{3}{4}\right) \qquad \gamma = \frac{\pi+\alpha}{2}$$
$$\beta = \delta - \alpha \qquad \delta = \frac{\pi-\alpha}{2}$$

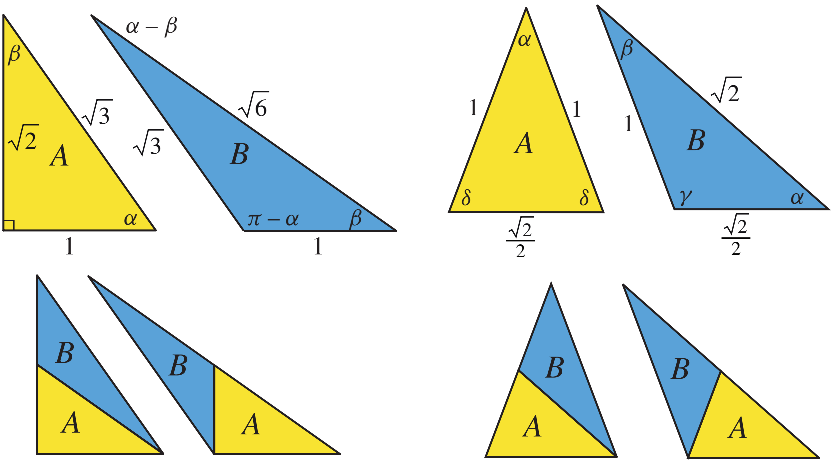


Figure 13 Two instances of the triangle pair in FIGURE 12 when one of them is right-angled (left) or isosceles (right)

In summary, given any parallelogram in which one diagonal is equal to $\sqrt{2}$ times the length of one side, then the same ratio obtains for the other diagonal, and the four triangles into which the two diagonals divide the parallelogram are similar to the two triangles into which one diagonal divides it, together with the two triangles into which the other diagonal divides it. In consequence, the two triangles corresponding to a and b (or c and d) in FIGURES 9, 10, or 11 will then furnish a pair of figures possessing the properties of a self-tiling tile set: a and b will together tile either c or d, which is to say, will tile larger versions of a or b themselves.

The properties of any such pair of triangles are summarized in FIGURE 12. Again, a couple of interesting cases occur when triangle A is either isosceles or right-angled, as seen in FIGURE 13. Whether there exist any order-2 sets different to those captured

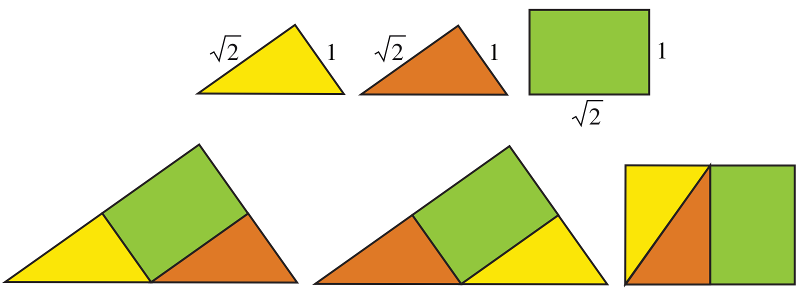


Figure 14 Frank Tinkelenberg's near solution using pieces of distinct area (and shape). The small triangles become doubled in size, while the larger rectangle is $\sqrt{2}$ times larger than the smaller. The latter can be replaced by any $1 \times \sqrt{2}$ parallelogram with appropriately modified triangles.

in the class of solutions here identified remains an unanswered question. Note that triangles A and B become congruent when $\alpha = 45°$.

Before closing, a point worth mentioning is the striking similarity to be found in comparing the triangle pairs considered above with another famous triangle duo known as the Robinson triangles. I refer here to the two "golden" triangles, first identified by Raphael Robinson [**7**], that result from bisecting the kites and darts familiar to us from Penrose tilings. (The triangles are also described by Penrose [**6**, pp. 32–37] and by Grünbaum and Shephard [**5**, Sec. 10.3].) Is there any significance to be read into the obvious correspondences between the two pairs? Alas, no. A simple procedure by means of which, given any triangle, a second can be constructed to result in a pair exhibiting analogous properties is easy to produce. Or in other words, in this respect the Robinson pair is merely one instance of an infinite family of such pairs, all of which are equally "significant."

Finally, I should like to say a word about the requirement included in the definition of a self-tiling tile set, the effect of which is to guarantee that the n enlarged (compound) copies of the n pieces are equal in size. It is the stipulation that *the scaling factor be the same for each piece.* For only thus does it follow that the n pieces must be of same area, a property that is assumed throughout the discussion above.

Nevertheless, although natural enough, such a demand is not essential. Consider an *unrestricted* self-tiling tile set, in which pieces may be of different sizes because the *scale* of the compound copies formed is permitted to differ from piece to piece. FIGURE 14 shows a near miss at such a solution using three pieces, due to Frank Tinkelenberg. Alas, two pieces are identical, despite which imperfection the use of pieces of distinct area is nicely demonstrated. Perhaps this example may encourage some readers to give thought to the unsolved problem of identifying a flawless self-tiling tile set of order 3, restricted or otherwise.

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Summary A novel species of self-similar tilings is introduced. Results divide naturally into two categories, involving either polyforms or non-polyforms. Some eye-catching findings are presented in what promises to be a rich field of research. The discovery of a class of triangle pairs showing arresting properties will undoubtedly surprise and appeal to many, not least those readers already conversant with rep-tiles.

LEE SALLOWS was born in 1944 and raised in post-war London. He has lived in Nijmegen in The Netherlands for the past forty-odd years. Until recently he worked as an electronics engineer for the Radboud University. A handful of published articles on computational wordplay and recreational mathematics are among the fruits of an idle, if occasionally inventive, life.