THE IMPOSSIBLE PROBLEM

A slip in the formulation of a near impossible puzzle made it actually unsolvable. Or did it?

by Lee Sallows

"*Miracles we perform instantly, the impossible may take a leetle longer*." (author's motto)

Truth is stranger than fiction, goes the saying, yet more often than not examples come down to us at second-hand: we read or hear of wonderful instances but seldom encounter the beast in our own forest. In the following I offer an unusual case, it is a true story that is stranger than fiction, but one that sceptical readers can put under a lens to examine, test, and verify themselves. Is any lesson to be drawn from this story? Judge for yourself.

Leafing through back numbers of *Scientific American* recently I came across an intriguing conundrum dubbed "The Impossible Problem" in Martin Gardner's *Mathmatical Games* department for December 1979. Then, as now, Gardner was the leading figure in recreational mathematics, his regular column famous as a trading centre in offbeat and exotic ideas. The Impossible Problem was new to me. "This beautiful problem," wrote Gardner, "I call ‘impossible’ because it seems to lack sufficient information for a solution". I could only agree: one reading and I was seriously hooked. "If there is a simpler solution than the one given, I should like to know about it," he wrote. Taking my severest thinking-cap from its hook in the hall and sinking into an armchair I surrendered myself to the challenge. Here is the problem exactly as Gardner presented it:

Two numbers (not necessarily different) are chosen from the range of positive integers greater than 1 and not greater than 20. Only the sum of the two numbers is given to mathematician S. Only the product of the two is given to mathematician P.

On the telephone S says to P: "I see no way you can determine my sum."

An hour later P calls him back to say: "I know your sum."

Later S calls P again to report: "Now I know your product."

What are the two numbers?

It took me four days to crack this nut. Halfway through I even wrote a computer program to assist the process. This was heavy handed, I admit, but the problem had got under my skin and after two days without a breakthrough desperation was setting in. Had Gardner not emphasized that the problem was virtually impossible I might have thrown in the towel; only his assurance that there *was* a solution kept me going. The computer print-out made it easier to survey relations among sums and products; it played no decisive role in cracking the problem but it did help to guide me toward a subtle insight that led to eventual victory. The problem had lived up to its name. Its solution was not only elegant, it called for some intricate thinking. Having triumphed at last, I carefully wrote out a description of the solution, double checked the result, and then reached for *Mathematical Games* to see how Gardner's approach compared. A surprise awaited me: his answer was *different* to mine.

I was less fazed by this than might be supposed. That a second answer based on a wholly different kind of argument might in principle exist had already crossed my mind. After all, the puzzle tells us a story about two people and some things they said to each other. Then we are asked, "What are the two numbers?" However, the two numbers referred to here never actually come into the story. What the question really boils down to is: Can you discover two numbers that *consistently explain* all the facts presented? The proposer's use of the definite article implies a unique solution, but given the flexible format, how could he ever be certain that a second was beyond devising? The discrepancy was thus explained: my own two numbers furnished a key that fitted the lock, Gardner could produce another pair that would open it as well. Even so, it seemed remarkable. To find the one solution had demanded hours of concentrated attack; the notion that an alternative existed strained credulity. Naturally I was more than a little curious to read his account. A glance showed me it took up a fair amount of space. Starting in however, I soon found myself baffled by his argument. As far as I could see it just didn't add up. Try as I might, I could not go along with his logic.

After a while I had an idea. Of course: it had to be an *error*. Gardner was bound to have published a correction in a subsequent column where all would be explained. I immediately began looking through *Mathematical Games* for the succeeding months. Sure enough, there it was in a postscript at the end of the column for March 1980: "As hundreds of readers have pointed out," I read, "the ‘impossible problem’ given in this department for December turned out .." *to contain an error in its solution*,” I filled in mentally. But I was wrong. Instead I read: ".. turned out to be literally impossible."

Literally impossible? I reeled. We had swung from one extreme to the other: one moment there are two solutions, the next none! "Because I gave an upper bound of 20 for the two selected numbers," he continued, "the solution became totally inapplicable." I thought this over and it began to make sense; this matter of the upper bound had been mentioned previously: "To simplify the problem I have given it here with an upper bound of 20 ... If you succeed in finding the unique solution[!], you will see how easily the problem can be extended by raising the upper bound. Surprisingly, if the bound is raised to 100, the answer remains the same." 100 had been its value in the original version of the problem as first described to him by a correspondent. Only now had he realized that it could not be reduced without incurring disaster. For example, at one stage in his solution the argument relies on eliminating certain sums that are expressible by different pairs of numbers, such as 35 = 16+19 = 4+31. Yet 31 is greater than 20, a contingency ruled out in his simplified version. In lowering the ceiling from 100 to 20 he had inadvertently made it impossible to eliminate these sums, and thus made it impossible to solve the problem.

Or so he thought. It was a natural assumption for one who believed the intended solution was unique. I had therefore discovered something that Martin Gardner never guessed. His Impossible Problem with its lower bound of 20 is *not* insoluble. But it *is* a tough cookie, in my estimation at least. Note carefully that I refer here to the problem exactly as reproduced above and not to any supposed equivalent or variation. In particular, the above should not be confused with its progenitor, the "same" problem that Gardner had received from a correspondent, the original publication of which he was able to announce later in a second postscript. In the sequel we shall see that in reworking this problem for presentation in *Mathematical Games*, Gardner changed more than the upper bound, but without ever realising that in so doing a new *kind* of solution became admitted.

Readers who enjoy a challenge may like to try their hand at the Impossible Problem before comparing notes with the solution detailed below. First however, since in certain very subtle points the statement of the puzzle lacks perfect clarity, let me add: (1) that *P* and *S* are indeed *aware* that the numbers they (simultaneously) receive are the product and sum of two integers greater than 1 and not greater than 20, (2) that each *knows* that the other is a mathematician, (3) that the statements they make are *true*, and are made in the *belief* that they are true by their speakers, and (4) that *P* and *S are* each seriously trying to discover the other's number and they announce their discovery of it just as soon as they succeed. These clarifications are entirely my own responsibility; Martin Gardner is blameless, although he would not demur on any particular, I feel sure. My intention is merely to dispel ambiguities that troubled me during my own assault on the problem, which is not to imply that the points raised necessarily play any part in the solution below. Prospective solvers should stop reading here. Good luck!

Exploring a Blind Alley

I said Gardner's changes in presentation had admitted a new solution. After studying the earlier form of the puzzle in the publication he cited I began to see that what he had really done was to inadvertently create a new problem. This explains why, even with the upper bound returned to 100, the solution he gives is still open to criticism. His solution is the correct answer to a subtly different problem. We shall look at this closely, and especially so since the initial step he takes is a very natural approach that still looks promising. It is an inference whose general validity is not affected by the value of the upper bound, and thus continues to offer a prospect of success in spite of its earlier failure. Nevertheless, in using it for a renewed attack on the Impossible Problem, we shall find that the argument is a diversion, a path that looks inviting but really leads nowhere. For all that, it is a passage in the labyrinth worth patiently exploring so that we shall know for sure that it does come to a dead end. Only then can we turn elsewhere with wholehearted confidence. Still later we shall see that the portal to this blind alley is actually a cunning aspect of the problem's "impossibility" since it serves as a decoy that distracts attention away from the real solution. Stranger still is that even this ingenious device is not the result of design, but merely another accidental feature of Gardner's fortuitous creation.

In the following I shall use *valnum* as shorthand for *valid number*, meaning any integer greater than 1 and not greater than 20. Let p stand for the product and s for the sum; x and y are the two unknown numbers. Here then is how Gardner opened his solution to the problem:

"After *S* said 'I see no way you can determine my sum,' *P* quickly realized that the sum cannot be the sum of two primes. To understand why, suppose the sum is 14. *S* would reason as follows: 'Perhaps the two numbers are the primes 3 and 11. Since their product, 33, has only the one pair of factors 3 and 11, *P* would know at once that my sum is 3 plus 11, or 14.' Therefore when *S* says *P* cannot know his sum, that tells *P* the sum cannot be the sum of two primes."

Armed with this insight Gardner goes on to eliminate many of the candidate values for s until he is left with ".. the seven possible sums: 11, 17, 23, 27, 29, 35 and 37." Focusing on each of these in turn, his subsequent arguments are able to disqualify all but one of these (assuming an upper bound of 100) to leave s = 17; x and y are eventually identified as 4 and 13.

sums	pairs	products
4	2,2	4
5	2,3	6
6	2,4	8
6	3,3	9
7	2,5	10
7	3,4	12 = 2×6
8	2,6	12 = 3×4
8	3,5	15
8	4,4	16 = 2×8
9	2,7	14
9	3,6	18 = 2×9
9	4,5	20 = 2×10
10	2,8	16 = 4×4
10	3,7	21
10	4,6	24 = 2×12 = 3×8
10	5,5	25
11	2,9	18 = 3×6
11	3,8	24 = 2×12 = 4×6
11	4,7	28 = 2×14
11	5,6	30 = 2×15 = 3×10
12	2,10	20 = 4×5
12	3,9	27
12	4,8	32 = 2×16
12	5,7	35
12	6,6	36 = 2×18 = 3×12 = 4×9
13	2,11	22
13	3,10	30 = 2×15 = 5×6
13	4,9	36 = 2×18 = 3×12 = 6×6
13	5,8	40 = 2×20 = 4×10
13	6,7	42 = 3×14
14	2,12	24 = 3×8 = 4×6
14	3,11	33
14	4,10	40 = 2×20 = 5×8
14	5,9	45 = 3×15
14	6,8	48 = 3×16 = 4×12
14	7,7	49
15	2,13	26
15	3,12	36 = 2×18 = 4×9 = 6×6
15	4,11	44
15	5,10	50
15	6,9	54 = 3×18
15	7,8	56 = 4×14
16	2,14	28 = 4×7
16	3,13	39
16	4,12	48 = 3×16 = 6×8
16	5,11	55
16	6,10	60 = 3×20 = 4×15 = 5×12
16	7,9	63
16	8,8	64 = 4×16
17	2,15	30 = 3×10 = 5×6
17	3,14	42 = 6×7
17	4,13	52
17	5,12	60 = 3×20 = 4×15 = 6×10
17	6,11	66
17	7,10	70 = 5×14
17	8,9	72 = 4×18 = 6×12
18	2,16	32 = 4×8
18	3,15	45 = 5×9
18	4,14	56 = 7×8
18	5,13	65
18	6,12	72 = 4×18 = 8×9
18	7,11	77
18	8,10	80 = 4×20 = 5×16
18	9,9	81
19	2,17	34
19	3,16	48 = 4×12 = 6×8
19	4,15	60 = 3×20 = 5×12 = 6×10
19	5,14	70 = 7×10
19	6,13	78
19	7,12	84 = 6×14
19	8,11	88
19	9,10	90 = 5×18 = 6×15
20	2,18	36 = 3×12 = 4×9 = 6×6
20	3,17	51
20	4,16	64 = 8×8
20	5,15	75
20	6,14	84 = 7×12
20	7,13	91
20	8,12	96 = 6×16
20	9,11	99
20	10,10	100 = 5×20
21	2,19	38
21	3,18	54 = 6×9
21	4,17	68
21	5,16	80 = 4×20 = 8×10
21	6,15	90 = 5×18 = 9×10
21	7,14	98
21	8,13	104
21	9,12	108 = 6×18
21	10,11	110
22	2,20	40 = 4×10 = 5×8
22	3,19	57
22	4,18	72 = 6×12 = 8×9
22	5,17	85
22	6,16	96 = 8×12
22	7,15	105
22	8,14	112 = 7×16
22	9,13	117
22	10,12	120 = 6×20 = 8×15
22	11,11	121
23	3,20	60 = 4×15 = 5×12 = 6×10
23	4,19	76
23	5,18	90 = 6×15 = 9×10
23	6,17	102
23	7,16	112 = 8×14
23	8,15	120 = 6×20 = 10×12
23	9,14	126 = 7×18
23	10,13	130
23	11,12	132
24	4,20	80 = 5×16 = 8×10
24	5,19	95
24	6,18	108 = 9×12
24	7,17	119
24	8,16	128
24	9,15	135
24	10,14	140 = 7×20
24	11,13	143
24	12,12	144 = 8×18 = 9×16
25	5,20	100 = 10×10
25	6,19	114
25	7,18	126 = 9×14
25	8,17	136
25	9,16	144 = 8×18 = 12×12
25	10,15	150
25	11,14	154
25	12,13	156
26	6,20	120 = 8×15 = 10×12
26	7,19	133
26	8,18	144 = 9×16 = 12×12
26	9,17	153
26	10,16	160 = 8×20
26	11,15	165
26	12,14	168
26	13,13	169
27	7,20	140 = 10×14
27	8,19	152
27	9,18	162
27	10,17	170
27	11,16	176
27	12,15	180 = 9×20 = 10×18
27	13,14	182
28	8,20	160 = 10×16
28	9,19	171
28	10,18	180 = 9×20 = 12×15
28	11,17	187
28	12,16	192
28	13,15	195
28	14,14	196
29	9,20	180 = 10×18 = 12×15
29	10,19	190
29	11,18	198
29	12,17	204
29	13,16	208
29	14,15	210
30	10,20	200
30	11,19	209
30	12,18	216
30	13,17	221
30	14,16	224
30	15,15	225
31	11,20	220
31	12,19	228
31	13,18	234
31	14,17	238
31	15,16	240 = 12×20
32	12,20	240 = 15×16
32	13,19	247
32	14,18	252
32	15,17	255
32	16,16	256
33	13,20	260
33	14,19	266
33	15,18	270
33	16,17	272
34	14,20	280
34	15,19	285
34	16,18	288
34	17,17	289
35	15,20	300
35	16,19	304
35	17,18	306
36	16,20	320
36	17,19	323
36	18,18	324
37	17,20	340
37	18,19	342
38	18,20	360
38	19,19	361
39	19,20	380
40	20,20	400

Table 1. The 40 possible sums and their associated products.

The tactic employed here may seem reasonable, but it neglects to exploit a significant improvement that can be made, particularly when the ceiling has been lowered to 20. To see how, suppose x and y were the *only* two valnums (not necessarily distinct) whose product is p. Cal l such a product *unique*. This may entail that x and y are both primes, but need not: 8 is unique since 2×4 is the only product of two valnums that will produce 8, and 4 is non-prime. So when Gardner points out that if P has a product with only one pair of factors then P could identify the sum, this applies equally to unique products in general, with or without two *prime* factors. Thus, by his same argument, when S says P cannot know his sum, that tells P the sum cannot be represented by any pair whose product is unique.

The effect of this modest refinement is crucial, a fact that became clear to me on looking over my computer print-out. For, as readers can easily verify, excepting a single case, every one of the possible sums from 4 to 40 can be formed by adding two valnums whose product is unique (see Table of Sums and Associated Products). In other words, we can now eliminate *six* of Gardner's seven possible sums. For example, 17 is 6+11, while $66=6\times11$ is unique. Thus, almost by accident, in one bound we have established that when P says "I know your sum," it must be because he knows that s is the sole remaining possibility: the number 11. This is the only sum between 4 and 40, *none* of whose possible summands, 2+9, or 3+8, or 4+7, or 5+6, multiply to produce a unique product: $2\times9=18=3\times6$, $3\times8=24=2\times12$, $4\times7=28=2\times14$, $5\times6=30=3\times10$.

So far so good. We have explained how P can deduce S's sum; the discrepancy between our 11 and Gardner's 17 is a result of his having overlooked the altered upper bound, although the precise detail of how this affects his argument need not detain us here. The only point remaining to account for is S's final statement: "Now I know your product." The trouble is, given 11, how *could S* ever discover p? As we have just seen, S would know that the product must be 18 or 24 or 28 or 30, but which is it? The more one considers his predicament the more irresolvable it seems. And the reason is simple: *S cannot* discover p. This is a fact we can prove.

The argument P uses to deduce that s is 11 is tedious to verify but pedestrian: 11 is the only integer between 4 and 40 that cannot be expressed as a sum of two valnums whose product is unique. Note that at no point does p come into it. P can deduce s is 11 without even knowing his own product. In fact, by using the same logic, S can *predict* that P can deduce s is 11, irrespective of the sum he may actually hold. What does this show? It shows that P's solitary statement *cannot* transmit any information about p to S. Hence, if S was unable to name the product before P spoke, neither will he be able to afterwards, which is what we set out to prove. Moreover, we have performed a *reductio ad absurdum*, for if our reasoning is correct then S's statement "Now I know your product," could never be true. Yet S's statements are true by definition. The argument that has brought us to this conclusion *must* therefore be invalid. What can be wrong with it?

Sherlock said it: "When you have eliminated the impossible, whatever remains, however improbable, must be the truth." Now the inference leading to our conclusion entails nothing but simple arithmetic, this we may safely eliminate. All that remains is the basic assumption: "After S said 'I see no way you can determine my sum,' P quickly realized that the sum cannot be a sum of two primes." [or "two valnums whose product is unique," in our extended version.]

Does it strike you that P is perhaps a bit *too* quick here? Recall that S and P are supposed to be mathematicians involved in a friendly competition to find the other's number. If not, then why not just *tell* each other their numbers? The assumption of a competitive element is essential to make sense of what happens. So, given a unique product, would P hesitate to factor his number, phone S and name his sum at once? No. Would S imagine that P would then hesitate? No. For P to infer as Gardner suggests, P's estimate of his opponent's mentality must be low indeed. At least, according to this view, P seems to think hat S is deliberately handing him a clue about his sum. He should be so lucky! What kind of an altruist does Gardner take *S* for? What kind of an optimist is *P* supposed to be? The more you look at it the more unrealistic it appears, and the same goes whatever the upper bound. This then is the premise upon which Gardner's opening argument is founded. How on earth did he hope to get away with it?

The explanation is simple: Gardner didn't know it, but he was giving us the solution to another problem—to the problem as it was before he changed its presentation! When applied to the original version his reasoning makes perfect sense, as we shall see. But in the meantime, *we* are still confronting The Impossible Problem as it *is* and no matter *how* it may have come into being, and in doing so we shall proceed on the usual assumption that it is a deliberately and carefully constructed puzzle. So to sum up: first we found that Gardner's (extended) argument would entail that the problem is insoluble because *S* could never have named the product. Next we saw that, although valid in other respects, this argument sets out from interpretations that conflict with common sense. Therefore, assuming the problem *is* solvable, there is no room left for any possible doubt: those interpretations of his *are* mistaken, his reading of *S*'s first statement *is* false. Relying on the old trick of persistently following the right hand wall, we have pursued this path through the labyrinth until it has returned us to our starting point. It is time for a fresh approach.

The Solution to the Problem

Happily, there is a simple alternative to all this. Pray take the basket chair. We have noted that if *P*'s product is unique he could have factored *p* and identified *x* and *y* immediately. But *P*, we are told, only deduces the sum an hour after hearing *S*'s first remark. So *p* must be the product of at least two distinct pairs of valnums, and *S*'s statement must convey some information that makes it possible for *P* to select the correct pair from among different candidates. Yet all *S* says is "I see no way you can determine my sum."

At first sight it is hard to see any useful information conveyed by this. What can *S*'s estimate of *P*'s ability to determine his sum communicate to *P* that he doesn't know already? Note however that the statement is made on the telephone. It may seem that the telephone is a mere incidental feature of the problem. However, an answer that can explain every detail of the situation is better than one that cannot. Thus, equipped with a telephone, *S* has had a chance to wait awhile before dialing *P*'s number. *P* might have called first, but didn't. Without the telephone as a giveaway, we might not have known that it was *possible* for *S* to pause and see whether *P* would respond quickly first.

In the meantime, while waiting, *S* could have listed each of the possible pairs of valnums whose sum is *s* and noted their corresponding products. The latter may *include* unique products, but reasoning as above, *S* will know that *p* cannot be one of these since otherwise *P* would have already phoned to say "I know your sum," or said the same right after he heard that it was *S* on the line. Hence *S*'s list must also contain one or more non-unique or *ambiguous* products, among them *p*. However, there is a special case to consider. For in the event that there were only *one* ambiguous product, *S* would then know that *it* had to be *p*! For example, suppose *s* is 7. The possible pairs of valnums that add to 7 are 2+5 and 3+4. Their corresponding products are 10 and 12. 10 is unique, while $12=2\times6=3\times4$ is ambiguous. Were *P*'s product 10 then *P* could work out in a flash that the two numbers are 2 and 5. But should *S* not hear from *P* fairly smartly then he will reason that *P* must have 12. A *solitary* ambiguous product on *S*'s list will always allow him to name *p*.

The question is though: *is S* able to name *p* at the time of his first call? The implication of what he says may or may not have been conciously intended by him, but is inescapable: *No*. For in *not* saying "I know your product," he reveals to *P* that he *cannot* yet name it, a fact subsequently confirmed by his second call: "*Now* I know your product." Of course, *P* might have concluded the same had *S* remained silent for long enough, but as it happens, *S* phones first. Until *S* speaks, for all *P* knows he could ring up at any moment to name the product. *S*'s first call will resolve *P*'s doubt.

Here then is a piece of *incidental* information conveyed by *S*'s initial remark, a tiny tidbit, but the key that we shall need. Granted that *S* might have said, "The walls are very perpendicular tonight," or almost anything else, and the tacit implication would have remained unchanged. Bear in mind, however, that any *irrelevant* remark would have alerted *P*, as it would have alerted *us*, that something surreptitious was afoot. As things stand, *S*'s remark *enables P* to infer something he didn't know before, while by the choice of words, "I see no way you can determine my sum," *we* have been sent off down the garden path on a wild goose chase through a blind alley in pursuit of a red herring. This statement is the decoy that leads us astray, the cunning device that says one thing while it means another. Put different words into *S*'s mouth and the problem becomes more tractable at the expense of its "impossibility". Does it not bear the very hallmark of Moriarty?

Thus, despite its unpromising appearance, *S*'s call has yielded a morsel of data for *P*. It is only a crumb, admittedly, but one that *P* might be able to use under special circumstances. For *P* can list the possible pairs of valnums whose product is *p* and note their corresponding sums. Taking each sum in turn, *P* can now put himself in *S*'s shoes and table what would then be *S*'s candidate products. Like us, *P* will have inferred that *S*'s actual list must show more than one ambiguous product. Were it the case that one, and only one, of *P*'s candidate sums gave rise to a list for *S* showing more than one ambiguous product then that sum would have to be *s*. Accordingly, our next question becomes: is there a product that could have placed *P* in this position?

P's product must lie between $2\times2=4$ and $20\times20=400$. Starting with the smallest, consider the possibilities in turn. Prime numbers and unique products can be ruled out, which disposes of 4,5,6,7,8,9,10 and 11. Next comes 12. If *P*'s product is 12 then *x* and *y* can only be 3 and 4, or 2 and 6. The corresponding sums are 7 and 8. We have just looked at the case when *S* has 7; it results in one ambiguous product. Thus, since *S*'s remark has shown that he cannot identify his product, *P* now knows that 7 is not the sum. But this would tell him that it *has* to be 8. Can it really be so?

We can check this against the foregoing. The pairs that sum to 8 are 2+6, 3+5, and 4+4. Their corresponding products are 12, 15, and 16. 15 is unique. But $12=2\times6=3\times4$ and $16= 4\times4=2\times8$ are ambiguous. 8 is thus the only one of *P*'s two candidate sums to give rise to a list for *S* showing more than one ambiguous product. It has worked exactly as predicted. We seem to have struck lucky amazingly quickly. Given 12, then once he knows that *S* cannot name his product, *P* can deduce that *S*'s sum is 8. It takes *P* an hour to do it, but then the underlying import of *S*'s remark will not have sunk in at once. Can *S* identify *P*'s product when given 8? No. It might be 12, it might be 16. Given 8, all he might do is to tease *P* with his seemingly innocuous, "I see no way you can determine my sum."

No way, that is, *until P* calls him back to say, "I know your sum" (it seems *P* didn't like being teased). For that would give *S* a fresh insight. Now *S* is a bright guy and quite capable of working out the foregoing chain f argument for himself. His discovery that a product of 12 is the only one of his two candidates, 12 and 16, that would have allowed *P* to name his sum is but a matter of time. At that point he phones *P* again to say, "Now I know your product." Everything is now explained. *P* has 12, *S* has 8, the two numbers are 2 and 6. Pray hand me my violin, Watson.

One solution is thus 2 and 6, but is this the only pair that works? We had hardly begun checking out *P*'s possible products; what happens beyond 12? An hour suffices to run through the remaining cases. My result can be checked by others. No further product will turn the same trick; 2 and 6 form the sole solution of its kind. However he did it, Martin Gardner has bequeathed us a gem.

Reconstructing the Crime

I have already said The Impossible Problem was the accidental fruit of changes Gardner introduced in presenting another problem. The time has come to examine this prodigy in detail. In the foregoing it has been convenient to speak of "Gardner's solution," but of course Gardner was merely reporting the known answer to that earlier problem. As his second postscript in *Mathematical Games* for May 1980 informs us, the earliest known appearance of the original problem is due to Hans Freudenthal, who presented it in *Nieuw Archief Voor Wiskunde* (Series 3, Vol. 17, 1969, p. 152). Two solutions received from readers, at root identical, were printed afterwards in the same Dutch journal (Vol. 18, 1970, pp. 102-6). What looks like an English translation of Freudenthal's problem then appeared six years later in *Mathematics Magazine* (Vol. 49, No.2, March 1976, p. 96), submitted by David J. Sprows. The solution given is again the same, the one that Gardner describes. The latter publication would seem to be the most likely source tapped by Mel Stover, the Winnipeg correspondent who brought it to Gardner's attention. A comparison between this and the *Mathematical Games* version of three years later reveals some interesting differences. Here is the Freudenthal/Sprows problem:

Let x and y be two numbers with $1<x<y$ and $x+y \leq 100$. Suppose S is given the value $x+y$ and P is given the value xy.
(1) *P* says: "I don't know the values of x and y."
(2) *S* replies: "I knew that you didn't know the values."
(3) *P* responds: "Oh, then I do know the values of x and y."
(4) *S* exclaims: "Oh, then so do I."
What are the values of x and y?

The close resemblance between this and The Impossible Problem is clear at a glance. The two, however, are distinct. We shall not examine the lengthy solution to this problem here, details of which can be found in the references cited, but press on with the comparison.

In the first place, Gardner does more than just *lower* the earlier upper bound. In the above problem the bound is defined differently: it is $x+y$ that must not exceed 100, so that y may range up to 98 as x falls to 2, their greatest value when equal then being 50. Notwithstanding, Gardner's definition seems to me the more natural, but what of his choice of 20? What would happen if the upper bound were changed? A computer program I wrote that is able to scan for solutions when different bounds are imposed has revealed a surprising fact: the solution of 2 and 6 is completely unaffected by the value of the bound, provided it is not less than eight. Whatever the ceiling value beyond this lower limit, 2 and 6 is always a solution, so that the very stipulation of an upper bound in the problem is completely superfluous, except in so far as it

Table 2. Solution pairs for different upper bounds (left) up to 100 (due to H. Diniz)

< 8 no solutions
8 - ((2 6) (4 6))
9 - ((2 6) (2 9))
10 - ((2 6) (5 8))
11 - ((2 6) (5 8))
12 - ((2 6) (8 9))
13 - ((2 6) (8 9))
14 - ((2 6) (7 12))
15 - ((2 6))
16 - ((2 6))
17 - ((2 6))
18 - ((2 6))
19 - ((2 6))
20 - ((2 6))
21 - ((2 6) (14 18))
22 - ((2 6) (11 16) (14 18))
23 - ((2 6) (11 16) (14 18))
24 - ((2 6))
25 - ((2 6) (16 18) (18 20))
26 - ((2 6) (18 20))
27 - ((2 6) (16 25))
28 - ((2 6) (16 27))
29 - ((2 6) (16 27))
30 - ((2 6))
31 - ((2 6))
32 - ((2 6) (20 30))
33 - ((2 6) (22 27))
34 - ((2 6) (22 27))
35 - ((2 6) (25 28))
36 - ((2 6) (24 30) (30 30))
37 - ((2 6) (24 30) (30 30))
38 - ((2 6) (30 30))
39 - ((2 6))
40 - ((2 6))
41 - ((2 6))
42 - ((2 6) (26 36) (25 42))
43 - ((2 6) (26 36) (25 42))
44 - ((2 6))
45 - ((2 6) (33 40))
46 - ((2 6) (33 40))
47 - ((2 6) (33 40))
48 - ((2 6) (36 36) (36 40))
49 - ((2 6) (36 36) (35 42))
50 - ((2 6) (35 42) (36 44) (40 42))
51 - ((2 6))
52 - ((2 6))
53 - ((2 6))
54 - ((2 6) (42 45))
55 - ((2 6) (44 45))
56 - ((2 6) (40 54))
57 - ((2 6) (36 56))
58 - ((2 6) (36 56))
59 - ((2 6) (36 56))
60 - ((2 6) (45 52) (48 50))
61 - ((2 6) (45 52) (48 50))
62 - ((2 6) (45 52) (48 50))
63 - ((2 6) (48 50))
64 - ((2 6) (48 56))
65 - ((2 6) (48 56))
66 - ((2 6) (48 63))
67 - ((2 6) (48 63))
68 - ((2 6))
69 - ((2 6))
70 - ((2 6) (55 56) (56 60))
71 - ((2 6) (55 56) (56 60))
72 - ((2 6) (49 72) (60 63))
73 - ((2 6) (49 72) (60 63))
74 - ((2 6) (49 72) (60 63))
75 - ((2 6) (60 63))
76 - ((2 6))
77 - ((2 6))
78 - ((2 6) (65 66))
79 - ((2 6) (65 66))
80 - ((2 6) (65 66) (65 72))
81 - ((2 6))
82 - ((2 6))
83 - ((2 6))
84 - ((2 6))
85 - ((2 6) (64 75) (72 77))
86 - ((2 6) (64 75) (72 77))
87 - ((2 6) (64 75) (72 77))
88 - ((2 6) (64 75) (60 88) (72 77))
89 - ((2 6) (64 75) (60 88) (72 77))
90 - ((2 6) (75 78))
91 - ((2 6))
92 - ((2 6) (72 80))
93 - ((2 6) (72 80))
94 - ((2 6) (72 80))
95 - ((2 6) (72 80))
96 - ((2 6) (72 92) (76 90))
97 - ((2 6) (72 92) (76 90))
98 - ((2 6) (72 92) (76 90))
99 - ((2 6) (72 92) (72 98))
100 - ((2 6) (84 88))

increases its difficulty through implying contingencies that do not exist. Thus, even Gardner's remark following the problem, where he says that the solution would remain good for every bound up to 100 and even beyond, turns out to be vindicated, as well.

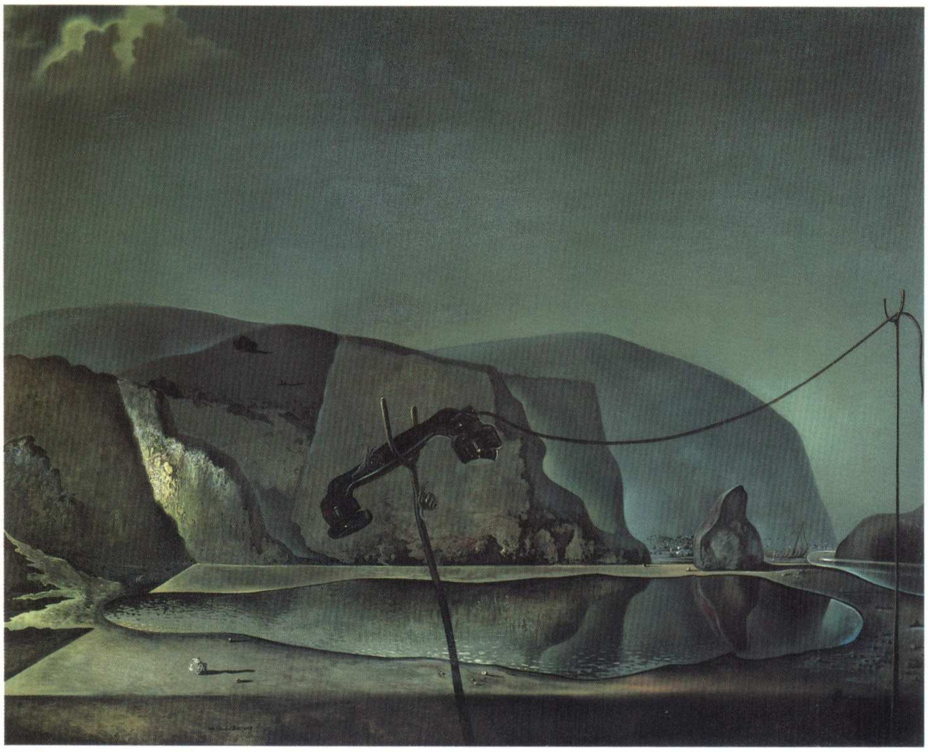
On the other hand, since products that are unique for one upper bound may become ambiguous with another, and vice versa, then depending on the bound in force, *extra* solutions can be created. In fact multiple solutions (up to seven) turn out to be the rule, as the Table of Solutions below will show. Even *lowering* the bound to certain values below 20 gives rise to more solutions. For instance, 8, the lowest bound to enable any solution, results in a second answer of 4 and 6, as readers can easily check. Similarly, bounds of 9,10,11,12, 13, and 14 also have two solutions, 15 has three, while beyond 20, 24 has five, 50 has six, 84 has seven, and so on. What distinguishes all these extra solutions from 2 and 6, however, is their bound-dependence; e.g., 5 and 8 are a solution when the upper bound is 10 or 11, but not for any other value. Most interesting of all though, is to discover that Gardner's choice of 20 is one among only fifteen upper bound values below 100 to result in a *unique* solution, which is of course in every case 2 and 6. Ironically, 100 is another instance, so luck was on his side again.

Secondly, unlike Gardner, Freudenthal/Sprows demand that the two numbers, x and y, be *distinct*. This is no trivial point. Had Gardner not added "not necessarily different," The Impossible Problem would have been killed at birth. This is because *S*'s sum, 8, could no longer be 4+4, which would leave 2+6 and 3+5 only, a change that disrupts our solution method. Curiously though, had Gardner asked for distinct x and y, but then used a different *upper bound*, the problem could have remained intact, as shown by Table 3 on page 6 below. For example, on including the condition $x \neq y$, the above mentioned program discovers that 15 is the first of thirty-nine values less than 100 to give rise to unique solutions. Starting with a lowest value of 10, we find 36 upper bounds that result in two or three different solutions, while 20 itself is one of just sixteen values greater than 10 that disrupt the problem. So once more, Gardner's decision was critical.

Thirdly, both the number of statements made and the order of the speakers in the two dialogs differ. Assuming the above *is* the text Gardner started with, we can imagine him thinking to himself that clarity would be gained by switching *S*'s first statement with *P*'s so as to rid the former of its retrospective stance. "I knew that you didn't know the values of x and y" would then become, "I see no way you can determine my sum." Knowing x and y is of course equivalent to knowing their sum. But having done this he will have seen that *P*'s statement (1) then becomes wholly redundant and can be dropped. The result is his *Mathematical Games* version using only three statements, which is admirably succinct.

Succinct yet *different*. The change looks harmless but is not. Starting from statements (1) and (2) above, Gardner's inference: "*P* quickly realized that the sum cannot be the sum of two primes," makes perfect sense. *S*'s "I *knew* .." reveals he was aware p could not be factored into two primes *before* deducing the same via statement (1), a conclusion he could only have arrived at from contemplating s alone. But in Gardner's new version, even when the upper bound is 100, the same inference is really a bit silly, since it overlooks the practical point that *P* would have named the sum first, had he been able to, while *S* is cast in the role of handing a hint to his opponent. Mathematicians tend to swallow this easily since their mode of thinking predisposes them to look through the words so as to fix on what they have already assumed are the mathematical essentials, a bold approach that affords no protection from booby traps! Moreover, for any number theorists familiar with Gardner's writings, the whole cast of his Impossible Problem is almost tantamount to a coded message saying: "This puzzle calls for some clever thinking involving prime numbers." The challenge as interpreted by the initiate is subtly different to the way it appears to the neophyte. Perhaps this explains why "hundreds of readers" wrote to him to point out the flaw in his answer, when, as we can now see, none of them could have tackled the problem seriously. In any case, Gardner's changes are again seen as crucial.

Fourthly, in a surreal move reminiscent of Salvador Dali, Gardner introduces a telephone into the landscape. Its role is twofold, I guess. It is a way of indicating that *S* and *P* are unable to see eachother's number, but it also tends to humanise the disembodied utterances of Freudenthal/Sprows' dialog, whose version was pitched at a mathematical audience, remember. Yet oh how snugly the telephone fits into the reconstruction of events as achieved in our new solution! There is *S* awaiting the call that will tell him *P* can name his sum. Time goes by and nothing happens. After concluding he cannot do it he decides to call him. Had this been cast in the disembodied utterence mode you could never be sure whether *S* had had an opportunity to wait for *P* to speak first. The telephone guarantees that opportunity. Gardner's telephone pours oil on the cogs of cognition as they grind toward a solution.



Mountain Lake 1938 by Salvador Dali

To conclude, therefore, four things distinguish Gardner's formulation of the problem from that of its original: the upper bound, the distinctness of x and y, the structure of the dialog, and the telephone. Not a one of these changes was necessitated; rather they are arbitrary, or the result of personal taste or whim. Overlooking for a moment what went wrong, certainly Gardner produced produced a crisper conundrum for his readers, but that might equally have been achieved in a hundred different ways. Coincidence is too weak a word to describe what has happened. It is almost as if some unseen force has guided the constructor's hand. Only the delicate combination of those particular changes he wrought have conspired to produce The Impossible Problem. Vary or omit but a single detail and the problem dissappears, or it cannot be solved, or it has too many solutions. Add to this that the new puzzle thus created, with its devilish decoy, the ulterior upper bound, and the sneaky significance of *S*'s remark, is itself even worthier than its prototype of the name "Impossible", and the whole series of events is revealed as nothing short of miraculous.

Were this a made up story the tale might please but would be dismissed with a smile. In the event, it is a true story that is stranger than fiction. Fun as unravelling it has been, I can only apologise to Martin Gardner, a long time mentor and idol of mine, for unearthing this skeleton from his closet and rattling it so loudly. Hopefully he will have been fascinated none the less.

The Superimpossible Problem

The Impossible Problem came about through chance. Could deliberate changes improve its formulation? Trying out various schemes, one thing led to another until I noticed that the principle at work in its solution can be extended to create a new problem that is even deeper. The result is quite amusing in that it presents a similar situation and demands a similar answer, while providing quite a bit *less* information than before. Indeed, The Superimpossible Problem, as it may aptly be called, appears so utterly incapable of solution as to suggest a joke. Need I emphasize, therefore, that the problem below certainly does provide sufficient information to reach an answer by means of straightforward reasoning without resort to any kind of hanky-panky? The solution, which will prove the point, follows immediately after, so that prospective solvers should be sure to cover it up now before reading further. Here is the problem:

A wealthy amateur mathematician, *A*, invited two eminent professionals, *P* and *S*, to take part in a competition for a large cash prize. Each knew the other's identity, but there was no prior contact between *P* and *S*. Seating them at separate tables divided by an intervening curtain, *A* addressed them from a central position visible to both. "I have here written on this piece of paper two distinct positive integers greater than two. Lying before you each is an envelope. Only the sum of these two integers is contained in your envelope, *S*, only their product is in your envelope, *P*. In a moment I shall give a sign, upon which you may both look at your numbers. The first of you who correctly names the two integers will receive a cash prize of $50,000, but I shall deduct $1,000 for every minute that elapses before you succeed. If the answer you give is wrong then the money you would have won will go to the other player. Pencils and paper are available on your tables, if required. That is all. Is everything quite clear?"

P and *S* nodded. After a moment *A* gave a sign and started a stopwatch. *P* and *S* then opened their envelopes simultaneously, drew out their numbers and began thinking. After about ten minutes *S* suddenly announced that he knew what the two integers were, and then named them. "That is the correct answer," responded *A* as he stopped his watch and held up the piece of paper to show the same two numbers *S* had named. *S* then received close on $40,000, as promised.

What were the two numbers?

Solution to the Superimpossible Problem

As previously, call the two unknown integers x and y, let p stand for the product and s for the sum; if there are only two distinct integers greater than two whose product is p, then call p *unique*. We can assume P and S did their utmost to answer the question as fast as possible because of the prize money diminishing rapidly with time. As eminent mathematicians their competence to reason and calculate fast is assured.

Suppose p is unique but not large, say less than a few hundred. P would then be able to factor his product with ease, and thus name x and y virtually at once. Hence, assuming a smallish sum, S, who could then see that p is not large, will know from P's silence that p cannot be unique. Similarly, suppose s is one of its two lowest possible values, 7 and 8, which can be formed only by 3+4 and 3+5 respectively. S would then be able to name x and y instantly instead of taking ten minutes. So s is greater than 8. Consider next the succeeding possibilities for s, in turn. The values involved are small, so that p would be small also. Observe that the time needed by P or S to deduce a fact should not be confused with how long it might take *us* to infer the same.

Suppose s=9, which can be formed by 3+6 or 4+5. Then p would be 18 or 20. But 18 and 20 are both unique products. Therefore s cannot be 9.

Suppose s=10, which is 3+7 or 4+6. Then p would be 21 or 24, the former unique but the latter non-unique: 24 = 3×8 or 4×6. S knows that if P had 21, he would be able to name x and y at once. Thus a silence of more than a minute would tell S he must have 24, and comparing this with s he would then know x and y at once. However, ten minutes elapse without S naming the two integers. Therefore s cannot be 10.

Suppose s=11, which is 3+8 or 4+7 or 5+6. Then p would be 24 or 28 or 30. Now 28 is unique, while 24 and 30 (= 3×10 or 5×6) are not. P's silence will have told S that p is not 28, but how could he decide between 24 and 30? S will consider P's situation.

Assume P has 24, which is 4×6 or 3×8. Then P would reason that s is 10 or 11. However, on considering 10, P would conclude as we have done above, and so realise that S could then name x and y within a couple of minutes at most. So, as time ticks by without S saying anything, after four or five minutes P could be certain that S must have 11 instead, and comparing this with his 24, he would then name x and y. Nevertheless, we know that P says nothing. Therefore p cannot be 24, and in the meantime, S, who will have been able to put himself in P's shoes and analyse the s=10 case himself, can deduce this exactly as we have.

Thus, with S holding 11, and having determined that p is not 24 or 28, he will know it can only be 30. Comparing p = 30 with s = 11, S now knows that x and y are 5 and 6. The complete argument has been intricate, however, and caution will demand a careful re-check before speaking, in case of any mistake. This will take but a few more minutes, bringing the total up to around ten, and then S would be utterly confident. At this point he names the two numbers as 5 and 6.

Can we be sure that this is the only answer of its kind? Imagine the competition had ended differently, with P naming the two integers after about five minutes. This is a new, simplified puzzle. On retracing the above solution, we can see that *P* must now have 24, while *S* still has 11; the two unknown integers are then 3 and 8. Does the logic involved ring a bell? The new puzzle is basically our old Impossible Problem with a few details changed: the upper bound is gone, the lower bound is now 3, not 2, and the two unknown numbers are defined as distinct. Earlier we noted that *P* would be able to tell that *S* cannot name his product were he to remain silent long enough, rather than speaking first at all. Hence *S*'s "I see no way you can determine my sum" can be excised, and still *P* would be able to say "I know your sum," after a short interval. The Superimpossible Problem arises from seeing what *S* could deduce were *P* then *not* to say anything after all: in the above case, that *P cannot* have 24, and thus must have 30, instead. The Superimpossible Problem is really a superstructure erected upon an underlying Impossible Problem. Hence the question: Can we be sure that 5 and 6 is the only solution of its kind?, devolves to a similar question about the uniqueness of the solution to the underlying Impossible Problem. The answer is yes, but I shall leave its proof with the reader.

Lastly, as I trust watchful readers will have noted, I have been careful not to assert here that the answers found to The Impossible and Superimpossible Problems are their *only* solutions, but merely the unique solutions *of their kind*. Might answers of a different kind exist? It seems pretty unlikely, but who will dare say? After all, as Watson replied to Holmes: "It is a wise man, Sherlock, who knows he has eliminated *everything* that is impossible."

Nijmegen, December, 2012

Table 3. Solution pairs for different upper bounds up to 100 when x is not equal to y. (due to H. Diniz)

<10 no solutions
10 - ((4 5) (3 8) (5 8))
11 - ((4 5) (3 8) (5 8))
12 - ((4 5) (4 9) (8 9))
13 - ((4 5) (4 9) (8 9))
14 - ((4 5) (7 12))
15 - ((4 5))
16 - NIL
17 - NIL
18 - ((9 12) (9 16))
19 - ((9 12) (9 16))
20 - NIL
21 - ((14 18))
22 - ((11 16) (14 18))
23 - ((11 16) (14 18))
24 - NIL
25 - ((16 18) (18 20))
26 - ((18 20))
27 - ((18 21))
28 - ((16 27))
29 - ((16 27))
30 - NIL
31 - NIL
32 - ((20 27))
33 - ((20 27))
34 - ((22 27))
35 - ((25 28))
36 - ((24 30) (27 32))
37 - ((24 30) (27 32))
38 - ((27 32))
39 - NIL
40 - NIL
41 - NIL
42 - ((26 36) (25 42))
43 - ((26 36) (25 42))
44 - NIL
45 - ((33 40))
46 - ((33 40))
47 - ((33 40))
48 - ((36 40))
49 - ((35 42))
50 - ((35 42))
51 - ((33 48))
52 - ((33 48) (40 45))
53 - ((33 48) (40 45))
54 - ((42 45))
55 - ((44 45))
56 - ((40 54))
57 - ((36 56))
58 - ((36 56))
59 - ((36 56))
60 - ((45 52) (48 50))
61 - ((45 52) (48 50))
62 - ((45 52) (48 50))
63 - ((48 50))
64 - ((48 56))
65 - ((48 56))
66 - ((48 63))
67 - ((48 63))
68 - ((48 65))
69 - NIL
70 - ((55 56) (56 60))
71 - ((55 56) (56 60))
72 - ((54 64) (49 72) (60 63))
73 - ((54 64) (49 72) (60 63))
74 - ((54 64) (49 72) (60 63))
75 - ((54 64) (60 63))
76 - ((54 64))
77 - ((54 64))
78 - ((65 66))
79 - ((65 66))
80 - ((65 66) (65 72))
81 - NIL
82 - NIL
83 - NIL
84 - NIL
85 - ((64 75) (72 77))
86 - ((64 75) (72 77))
87 - ((64 75) (68 75) (72 77))
88 - ((64 75) (60 88) (72 77))
89 - ((64 75) (60 88) (72 77))
90 - ((75 78))
91 - NIL
92 - ((72 80))
93 - ((72 80))
94 - ((72 80))
95 - ((72 80))
96 - ((72 92) (76 90))
97 - ((72 92) (76 90))
98 - ((72 92) (76 90))
99 - ((72 92) (75 96))
100 - ((80 85) (84 88))